

Affine connections on involutive G -structures

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0. Introduction. An affine connection is one of the central objects in differential geometry. One of its most informative characteristics is the (restricted) holonomy group which is defined, up to a conjugation, as a subgroup of $GL(T_t M)$ consisting of all automorphisms of the tangent space $T_t M$ at a point $t \in M$ induced by parallel translations along the t -based contractible loops in M . The list of groups which can be holonomies of affine connections is disappointingly dull — according to Hano and Ozeki [H-O], *any* closed subgroup of a general linear group can be realized in this way. The situation, however, is very different in the subclass of affine connections with *zero torsion*. Long ago, Berger [Be] presented a very restricted list of possible irreducibly acting holonomies of torsion-free affine connections. His list was complete in the part of *metric* connections (and later much work has been done to refine this "metric" part of his list, see, e.g., [Br1] and references cited therein), while the situation with holonomies of *non-metric* torsion-free affine connections was and remains very unclear. One of the results that will be discussed in this paper asserts that *any* torsion-free holomorphic affine connection with irreducibly acting holonomy group can, in principle, be constructed by twistor methods. Another result reveals a new natural subclass of affine connections with *very little torsion* which shares with the class of torsion-free affine connections two basic properties — the list of irreducibly acting holonomy groups of affine connections in this subclass is very restricted and the links with the twistor theory are again very strong.

The purpose of this paper is to explain the key elements of the above mentioned twistor constructions without indulging in rather lengthy proofs. We work throughout in the category of complex manifolds, holomorphic affine connections, etc., though many results can be easily adapted to the real analytic case along the lines explained in [M].

1. Irreducible G -structures. When studying an affine connection ∇ with the irreducibly acting holonomy group G , it is suitable to work with the associated G -structure. In this section we recall some notions of the theory of G -structures.

Let M be an m -dimensional complex manifold and $\mathcal{L}^* M$ the holomorphic coframe bundle $\pi : \mathcal{L}^* M \rightarrow M$ whose fibres $\pi^{-1}(t)$ consist of all \mathbb{C} -linear isomorphisms $e : \mathbb{C}^m \rightarrow \Omega_t^1 M$, where $\Omega_t^1 M$ is the cotangent space at $t \in M$. The space $\mathcal{L}^* M$ is a principle right $GL(m, \mathbb{C})$ -bundle with the right action given by $R_g(e) = e \circ g$. If G is a closed subgroup of $GL(m, \mathbb{C})$, then a (holomorphic) G -structure on M is a principal subbundle \mathcal{G} of $\mathcal{L}^* M$ with the group G . It is clear that there is a one-to-one correspondence between the set of G -structures on M and the set of holomorphic sections σ of the quotient bundle $\tilde{\pi} : \mathcal{L}^* M / G \rightarrow M$ whose typical fibre is isomorphic to $GL(m, \mathbb{C}) / G$. A G -structure on M is called *locally flat* if there exists a coordinate patch in the neighbourhood of each point $t \in M$

such that in the associated canonical trivialization of \mathcal{L}^*M/G over this patch the section σ is represented by a constant $GL(m, \mathbb{C})/G$ -valued function. A G -structure is called *k-flat* if, for each $t \in M$, the k -jet of the associated section σ of \mathcal{L}^*M/G at t is isomorphic to the k -jet of some locally flat section of \mathcal{L}^*M/G . It is not difficult to show that a G -structure admits a torsion-free affine connection if and only if it is 1-flat (cf. [Br1]). A G -structure on M is called *irreducible* if the action of G on \mathbb{C}^m leaves no non-zero invariant subspaces.

Given an affine connection ∇ on a connected simply connected complex manifold M with the irreducibly acting holonomy group G , the associated irreducible G -structure $\mathcal{G}_\nabla \subset \mathcal{L}^*M$ can be constructed as follows. Define two points u and v of \mathcal{L}^*M to be equivalent, $u \sim v$, if there is a holomorphic path γ in M from $\pi(u)$ to $\pi(v)$ such that $u = P_\gamma(v)$, where $P_\gamma : \Omega_{\pi(v)}^1 M \rightarrow \Omega_{\pi(u)}^1 M$ is the parallel transport along γ . Then \mathcal{G}_∇ can be defined, up to an isomorphism, as $\{u \in \mathcal{L}^*M \mid u \sim v\}$ for some coframe v . The G -structure \mathcal{G}_∇ is the smallest subbundle of \mathcal{L}^*M which is invariant under ∇ -parallel translations.

It will be shown later that for any holomorphic irreducible G -structure $\mathcal{G} \rightarrow M$ there is associated an analytic family of compact isotropic submanifolds $\{X_t \hookrightarrow Y \mid t \in M\}$ of a certain complex contact manifold Y which encodes much information about \mathcal{G} . To explain this correspondence in more detail, we first digress in the next two sections to the Kodaira [K] deformation theory of compact complex submanifolds and to its particular generalization studied in [Me1].

2. Kodaira relative deformation theory. Let Y and M be complex manifolds and let $\pi_1 : Y \times M \rightarrow Y$ and $\pi_2 : Y \times M \rightarrow M$ be natural projections. An analytic family of compact submanifolds of the complex manifold Y with the parameter space M is a complex submanifold $F \hookrightarrow Y \times M$ such that the restriction of the projection π_2 on F is a proper regular map (regularity means that the rank of the differential of $\nu := \pi_2|_F : F \rightarrow M$ is equal to $\dim M$ at every point). The parameter space M is called a *Kodaira moduli space*. Thus the family F has a double fibration structure

$$Y \xleftarrow{\mu} F \xrightarrow{\nu} M$$

where $\mu := \pi_1|_F$. For each $t \in M$ we say that the compact complex submanifold $X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y$ belongs to the family F . Sometimes we use a more explicit notation $\{X_t \hookrightarrow Y \mid t \in M\}$ to denote an analytic family F of compact submanifolds.

If $F \hookrightarrow Y \times M$ is an analytic family of compact submanifolds, then, for any $t \in M$, there is a natural linear map [K],

$$k_t : T_t M \longrightarrow H^0(\nu^{-1}(t), N_{\nu^{-1}(t)|F}) \xrightarrow{\mu_*} H^0(X_t, N_{X_t|Y}),$$

which is a composition of the natural lift of a tangent vector at t to a global section of the normal bundle of the submanifold $\nu^{-1}(t) \hookrightarrow F$ with the Jacobian of μ (here the symbol $N_{A|B}$ stands for the normal bundle of a submanifold $A \hookrightarrow B$). An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds is called *complete* if the map k_t is an isomorphism for all $t \in M$ which in particular implies that $\dim M = h^0(X_t, N_{X_t|Y})$.

In 1962 Kodaira [K] proved the following existence theorem: *if $X \hookrightarrow Y$ is a compact complex submanifold with normal bundle N such that $H^1(X, N) = 0$, then X belongs to a complete analytic family $F \hookrightarrow Y \times M$ of compact submanifolds of Y .*

3. Deformations of compact complex Legendre submanifolds of complex contact manifolds.

In this section we shall be interested in the following specialisation (which will eventually turn out to be a generalisation) of the Kodaira relative deformation problem: the initial data is a pair $X \hookrightarrow Y$ consisting of a compact complex Legendre submanifold X of a complex contact manifold Y and the object of study is the set, M , of all holomorphic deformations of X inside Y which remain Legendre. First, we recall some standard notions, then give a better formulation of the problem, and finally present its solution.

Let Y be a complex $(2n + 1)$ -dimensional manifold. A complex contact structure on Y is a rank $2n$ holomorphic subbundle $D \subset TY$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\begin{aligned} \Phi : D \times D &\longrightarrow TY/D \\ (v, w) &\longrightarrow [v, w] \bmod D \end{aligned}$$

is non-degenerate. Define the contact line bundle L by the exact sequence

$$0 \longrightarrow D \longrightarrow TY \xrightarrow{\theta} L \longrightarrow 0.$$

One can easily verify that maximal non-degeneracy of the distribution D is equivalent to the fact that the above defined "twisted" 1-form $\theta \in H^0(Y, L \otimes \Omega^1 M)$ satisfies the condition $\theta \wedge (d\theta)^n \neq 0$. A complex submanifold $X \hookrightarrow Y$ is called *isotropic* if $TX \subset D$. An isotropic submanifold of maximal possible dimension n is called *Legendre*. In this paper we shall be primarily interested in compact Legendre submanifolds. The normal bundle $N_{X|Y}$ of any Legendre submanifold $X \hookrightarrow Y$ is isomorphic to $J^1 L_X$ [L2], where $L_X = L|_X$. Therefore, $N_{X|Y}$ fits into the exact sequence

$$0 \longrightarrow \Omega^1 X \otimes L_X \longrightarrow N_{X|Y} \xrightarrow{pr} L_X \longrightarrow 0.$$

Let Y be a complex contact manifold. An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds of Y is called an *analytic family of compact Legendre submanifolds* if, for any point $t \in M$, the corresponding subset $X_t := \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is a Legendre submanifold. The parameter space M is called a *Legendre moduli space*.

Let $F \hookrightarrow Y \times M$ be an analytic family of compact Legendre submanifolds. According to Kodaira [K], there is a natural linear map $k_t : T_t M \longrightarrow H^0(X_t, N_{X_t|Y})$. We say that the family F is *complete at a point* $t \in M$ if the composition

$$s_t : T_t M \xrightarrow{k_t} H^0(X_t, N_{X_t|Y}) \xrightarrow{pr} H^0(X_t, L_{X_t})$$

provides an isomorphism between the tangent space to M at the point t and the vector space of global sections of the contact line bundle over X_t . One of the motivations behind this definition is the fact [Me1] that an analytic family of compact Legendre submanifolds $\{X_t \hookrightarrow Y \mid t \in M\}$ which is complete at a point $t_0 \in M$ is also *maximal* at t_0 in the sense that, for any other analytic family of compact Legendre submanifolds $\{X_{\tilde{t}} \hookrightarrow Y \mid \tilde{t} \in \tilde{M}\}$ such that $X_{t_0} = X_{\tilde{t}_0}$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists a neighbourhood $\tilde{U} \subset \tilde{M}$ of \tilde{t}_0 and a holomorphic map $f : \tilde{U} \longrightarrow M$ such that $f(\tilde{t}_0) = t_0$ and $X_{f(\tilde{t}')} = X_{\tilde{t}'}$ for each $\tilde{t}' \in \tilde{U}$. An analytic family $F \hookrightarrow Y \times M$ is called *complete* if it is complete at each point of the Legendre moduli space M . In this case M is also called complete.

The following result [Me1] reveals a simple condition for the existence of complete Legendre moduli spaces.

Theorem 1 *Let X be a compact complex Legendre submanifold of a complex contact manifold (Y, L) . If $H^1(X, L_X) = 0$, then there exists a complete analytic family of compact Legendre submanifolds $F \hookrightarrow Y \times M$ containing X . This family is maximal and $\dim M = h^0(X, L_X)$.*

Let X be a complex manifold and L_X a line bundle on X . There is a natural "evaluation" map $H^0(X, L_X) \otimes \mathcal{O}_X \longrightarrow J^1 L_X$ whose dualization gives rise to the canonical map

$$L_X \otimes S^{k+1}(J^1 L_X)^* \longrightarrow L_X \otimes S^k(J^1 L_X)^* \otimes [H^0(X, L_X)]^*$$

which in turn gives rise to the map of cohomology groups

$$H^1(X, L_X \otimes S^{k+1}(J^1 L_X)^*) \xrightarrow{\phi} H^1(X, L_X \otimes S^k(J^1 L_X)^*) \otimes [H^0(X, L_X)]^*.$$

For future reference, we define a vector subspace

$$\tilde{H}^1(X, L_X \otimes S^{k+1}(J^1 L_X)^*) := \ker \phi \subset H^1(X, L_X \otimes S^{k+1}(J^1 L_X)^*).$$

4. G -structures induced on Legendre moduli spaces of generalized flag varieties.

Recall that a generalised flag variety X is a compact simply connected homogeneous Kähler manifold [B-E]. Any such a manifold is of the form $X = G/P$, where G is a complex semisimple Lie group and $P \subset G$ a fixed parabolic subgroup. Assume that such an X is embedded as a Legendre submanifold into a complex contact manifold (Y, L) with contact line bundle L such that $L_X := L|_X$ is very ample. Then the Bott-Borel-Weil theorem and the fact that any holomorphic line bundle on X is homogeneous imply that $H^1(X, L_X) = 0$. Therefore, by Theorem 1, there exists a complete analytic family of compact Legendre submanifolds $\{X_t \hookrightarrow Y \mid t \in M\}$, i.e. the initial data " $X \hookrightarrow Y$ " give rise to a new complex manifold M which, as the following result shows, comes equipped with a rich geometric structure.

Theorem 2 [Me1] *Let X be a generalised flag variety embedded as a Legendre submanifold into a complex contact manifold Y with contact line bundle L such that L_X is very ample on X . Then*

- (i) *There exists a complete analytic family $F \hookrightarrow Y \times M$ of compact Legendre submanifolds with moduli space M being an $h^0(X, L_X)$ -dimensional complex manifold. For each $t \in M$, the associated Legendre submanifold X_t is isomorphic to X .*
- (ii) *The Legendre moduli space M comes equipped with an induced irreducible G -structure, $\mathcal{G}_{ind} \rightarrow M$, with G isomorphic to the connected component of the identity of the group of all global biholomorphisms $\phi : L_X \rightarrow L_X$ which commute with the projection $\pi : L_X \rightarrow X$. The Lie algebra of G is isomorphic to $H^0(X, L_X \otimes (J^1 L_X)^*)$.*
- (iii) *If \mathcal{G}_{ind} is k -flat, $k \geq 0$, then the obstruction for \mathcal{G}_{ind} to be $(k+1)$ -flat is given by a tensor field on M whose value at each $t \in M$ is represented by a cohomology class $\rho_t^{[k+1]} \in \tilde{H}^1(X_t, L_{X_t} \otimes S^{k+2}(J^1 L_{X_t})^*)$.*
- (iv) *If \mathcal{G}_{ind} is 1-flat, then the bundle of all torsion-free connections in \mathcal{G}_{ind} has as the typical fiber an affine space modeled on $H^0(X, L_X \otimes S^2(J^1 L_X)^*)$.*

REMARK. Theorem 2 is actually valid for a larger class of compact complex manifolds X than the class of generalized flag varieties — the only vital assumptions are [Me1] that X is rigid and the cohomology groups $H^1(X, \mathcal{O}_X)$ and $H^1(X, L_X)$ vanish.

The geometric meaning of cohomology classes $\rho_t^{[k+1]} \in \tilde{H}^1(X_t, L_{X_t} \otimes S^{k+2}(J^1 L_{X_t})^*)$ of Theorem 2(iii) is very simple — they compare to $(k+2)$ th order the germ of the Legendre embedding $X_t \hookrightarrow Y$ with the "flat" model, $X_t \hookrightarrow J^1 L_{X_t}$, where the ambient contact manifold is just the total space of the vector bundle $J^1 L_{X_t}$ together with its canonical contact structure and the Legendre submanifold X_t is realised as a zero section of $J^1 L_{X_t} \rightarrow X_t$. Therefore, the cohomology class $\rho_t^{[k]}$ can be called the k th Legendre jet of X_t in Y . Then it is natural to call a Legendre submanifold $X_t \hookrightarrow Y$ k -flat if $\rho_t^{[k]} = 0$. With this terminology, the item (iii) of Theorem 2 acquires a rather symmetric form: *the induced G -structure on the moduli space M of a complete analytic family of compact Legendre submanifolds is k -flat if and only if the family consists of k -flat Legendre submanifolds.*

This general construction can be illustrated by three well known examples which were among the motivations behind the present work (in fact the list of examples can be made much larger — Theorem 2 has been checked for all "classical" torsion-free geometries as well as for a large class of locally symmetric structures). The first example is a "generic" $GL(m, \mathbb{C})$ -structure on an m -dimensional manifold M . The associated twistorial data $X \hookrightarrow Y$ is easy to describe: the complex contact manifold Y is the projectivized cotangent bundle $\mathbb{P}(\Omega^1 M)$ with its natural contact structure while $X = \mathbb{CP}^{m-1}$ is just a fiber of the projection $\mathbb{P}(\Omega^1 M) \rightarrow M$. The corresponding complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ is the set of all fibres of this fibration. Since $L_X = \mathcal{O}(1)$ and $J^1 L_X = \mathbb{C}^m \otimes \mathcal{O}_X$, we have $H^1(X, L_X \otimes S^{k+2}((J^1 L_X)^*)) = 0$ for all $k \geq 0$ which confirms the well-known fact that any $GL(m, \mathbb{C})$ -structures on an m -dimensional manifold are locally flat.

The second example [L1] is a pair $X \hookrightarrow Y$ consisting of an n -quadric Q_n embedded into a $(2n+1)$ -dimensional contact manifold (Y, L) with $L|_X \simeq i^* \mathcal{O}_{\mathbb{CP}^{n+1}}(1)$, $i : Q_n \rightarrow \mathbb{CP}^{n+1}$ being a standard projective realisation of Q_n . It is easy to check that in this case $H^0(X, L_X \otimes (J^1 L_X)^*)$ is precisely the conformal algebra implying that the associated $(n+2)$ -dimensional Legendre moduli space M comes equipped canonically with a conformal structure. Since $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$, the induced conformal structure must be torsion-free in agreement with the classical result of differential geometry. Easy calculations show that the vector space $H^1(X, L_X \otimes S^3(J^1 L_X)^*)$ is exactly the subspace of $TM \otimes \Omega^1 M \otimes \Omega^2 M$ consisting of tensors with Weyl curvature symmetries. Thus Theorem 2(iii) implies the well-known Schouten conformal flatness criterion. Since $H^0(X, L_X \otimes S^2(J^1 L_X)^*)$ is isomorphic to the typical fibre of $\Omega^1 M$, the set of all torsion-free affine connections preserving the induced conformal structure is the affine space modeled on $H^0(M, \Omega^1 M)$, again in agreement with the classical result.

The third example is Bryant's [Br2] relative deformation problem $X \hookrightarrow Y$ with X being a rational Legendre curve \mathbb{CP}^1 in a complex contact 3-fold (Y, L) with $L_X = \mathcal{O}(3)$. Calculating $H^0(X, L_X \otimes (J^1 L_X)^*)$, one easily concludes that the induced G -structure on the associated 4-dimensional Legendre moduli space is exactly an exotic G_3 -structure which has been studied by Bryant in his search for irreducibly acting holonomy groups of torsion-free affine connections which are missing in the Berger list [Be] (the missing holonomies are called *exotic*). Since $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$, Theorem 2(iii) says that the induced G_3 -structure is torsion-free in accordance with [Br2]. Since $H^0(X, L_X \otimes S^2(J^1 L_X)^*) = 0$, \mathcal{G}_{ind} admits a unique torsion-free affine connection ∇ . The cohomology class $\rho_t^{[2]} \in$

$H^1(X, L_X \otimes S^3(J^1 L_X)^*)$ from Theorem 2(iii) is exactly the curvature tensor of ∇ .

How large is the family of G -structures which can be constructed by twistor methods of Theorem 2? As the following result [Me1] shows, in the category of irreducible 1-flat G -structures this class is as large as one could wish.

Theorem 3 (i) *Let H be one of the following representations: (a) $Spin(2n+1, \mathbb{C})$ acting on \mathbb{C}^{2n} , $n \geq 3$; (b) $Sp(2n, \mathbb{C})$ acting on \mathbb{C}^{2n} , $n \geq 2$; (c) G_2 acting on \mathbb{C}^7 . Suppose that $G \subset GL(m, \mathbb{C})$ is a connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups contains H . If \mathcal{G} is any irreducible 1-flat $G \cdot \mathbb{C}^*$ -structure on an m -dimensional manifold M , then there exists a complex contact manifold (Y, L) and a generalised flag variety X embedded into Y as a Legendre submanifold with L_X being very ample, such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} \subset \mathcal{G}_{ind}$. In particular, when $G = H$ one has in the case (a) $X = SO(2n+2, \mathbb{C})/U(n+1)$ and \mathcal{G}_{ind} is a $Spin(2n+2, \mathbb{C}) \cdot \mathbb{C}^*$ -structure; in the case (b) $X = \mathbb{CP}^{2n-1}$ and \mathcal{G}_{ind} is a $GL(2n, \mathbb{C})$ -structure; and in the case (c) $X = Q_5$ and \mathcal{G}_{ind} is a $CO(7, \mathbb{C})$ -structure.*

(ii) *Let $G \subset GL(m, \mathbb{C})$ be an arbitrary connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups does not contain any of the groups H considered in (i). If \mathcal{G} is any irreducible 1-flat $G \cdot \mathbb{C}^*$ -structure on an m -dimensional manifold M , then there exists a complex contact manifold (Y, L) and a Legendre submanifold $X \hookrightarrow Y$ with $X = G/P$ for some parabolic subgroup $P \subset G$ and with L_X being very ample, such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} = \mathcal{G}_{ind}$.*

The conclusion is that there are very few irreducible G -structures which can *not* be constructed by twistor methods discussed in this paper. It is also worth pointing out that Theorem 2(iii) gives rise to a new and rather effective machinery to search for exotic holonomies. The new results in this direction will be discussed elsewhere — here we only note that the claimed efficiency of the twistor technique is largely due to the simple observation that the key cohomology groups $H^1(X_t, L_{X_t} \otimes S^2(J^1 L_{X_t})^*)$ and $H^1(X_t, L_{X_t} \otimes S^3(J^1 L_{X_t})^*)$, which provide us with the full information about torsion and curvature tensors, can be computed by a *combination* of the representation theory methods (such as Bott-Borel-Weil theorem) and the methods of complex analysis. In some important cases it is even enough to use the complex analysis methods only.

5. Torsion-free affine connections. Let $F \hookrightarrow Y \times M$ be a complete analytic family of compact Legendre submanifolds. Any point t in M is represented thus by a compact complex Legendre submanifold X_t . The first floors of the two towers of infinitesimal neighbourhoods of the analytic spaces $t \hookrightarrow M$ and $X_t \hookrightarrow Y$ are related to each other via the isomorphism $T_t M = H^0(X_t, L_{X_t})$. What happens at the second floors of these two towers? If $J_t \subset \mathcal{O}_M$ is the ideal of holomorphic functions which vanish at $t \in M$, then the tangent space $T_t M$ is isomorphic to $(J_t/J_t^2)^*$. Define a second order tangent bundle, $T_t^{[2]} M$, at the point t as $(J_t/J_t^3)^*$. Then, evidently, $T_t^{[2]} M$ fits into an exact sequence of complex vector spaces

$$0 \longrightarrow T_t M \longrightarrow T_t^{[2]} M \longrightarrow S^2(T_t M) \longrightarrow 0 \quad (1)$$

For each $t \in M$ there exists a holomorphic vector bundle, $\Delta_{X_t}^{[2]}$, on the associated Legendre submanifold $X_t \hookrightarrow Y$ such that there are an exact sequence of locally free sheaves

$$0 \longrightarrow L_{X_t} \xrightarrow{\alpha} \Delta_{X_t}^{[2]} \longrightarrow S^2(J^1 L_{X_t}) \longrightarrow 0 \quad (2)$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_t M & \longrightarrow & T_t^{[2]} M & \longrightarrow & S^2(T_t M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X_t, L_{X_t}) & \longrightarrow & H^0(X_t, \Delta_{X_t}^{[2]}) & \longrightarrow & H^0(X_t, S^2(J^1 L_{X_t})) & \longrightarrow & 0 \end{array} \quad (3)$$

which extends the canonical isomorphism $T_t M \rightarrow H^0(X_t, L_{X_t})$ to *second* order infinitesimal neighbourhoods of $t \hookrightarrow M$ and $X_t \hookrightarrow Y$. For the details of the construction of $\Delta_{X_t}^{[2]}$ we refer the interested reader to [Me1]. In this paper we need only to know that this bundle exists and has the stated properties. The extension (2) defines a cohomology class

$$\rho_t^{[1]} \in \text{Ext}_{\mathcal{O}_{X_t}}^1(S^2(J^1 L_{X_t}), L_{X_t}) = H^1(X_t, L_{X_t} \otimes S^2(J^1 L_{X_t})^*).$$

This is exactly the class of Theorem 2(iii) which is the obstruction to 1-flatness of X_t in Y . Therefore, if X_t is 1-flat, then extension (2) splits, i.e. there exists a morphism $\beta : \Delta_{X_t}^{[2]} \rightarrow L_{X_t}$ such that $\beta \circ \alpha = \text{id}$. Any such a morphism induces via the commutative diagram (3) an associated splitting of the exact sequence (1) which is equivalent to a torsion-free affine connection at $t \in M$. A torsion-free connection on the Legendre moduli space M which arises at each $t \in M$ from a splitting of the extension (2) is called an *induced* connection. Now we can formulate the main theorem about torsion-free affine connections.

Theorem 4 [Me1] *Let ∇ be a holomorphic torsion-free affine connection on a complex manifold M with irreducibly acting reductive holonomy group G . Then there exists a complex contact manifold (Y, L) and a 1-flat Legendre submanifold $X \hookrightarrow Y$ with $X = G_s/P$ for some parabolic subgroup P of the semisimple factor G_s of G and with L_X being very ample, such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and ∇ is an induced torsion-free affine connection in \mathcal{G}_{ind} .*

The conclusion is that any holomorphic torsion-free affine connection with irreducibly acting holonomy group can, in principle, be constructed by twistor methods.

6. From Kodaira to Legendre moduli spaces and back. In this subsection we first show that *any* complete Kodaira moduli space can be interpreted as a complete Legendre moduli space and then use this fact to prove a proposition about canonically induced geometric structures on Kodaira moduli spaces.

If $X \hookrightarrow Y$ is a complex submanifold, there is an exact sequence of vector bundles

$$0 \longrightarrow N_{X|Y}^* \longrightarrow \Omega^1 Y|_X \longrightarrow \Omega^1 X \longrightarrow 0,$$

which induces a natural embedding, $\mathbb{P}(N_{X|Y}^*) \hookrightarrow \mathbb{P}(\Omega^1 Y)$, of total spaces of the associated projectivised bundles. The manifold $\hat{Y} = \mathbb{P}(\Omega^1 Y)$ carries a natural contact structure such that the constructed embedding $\hat{X} = \mathbb{P}(N_{X|Y}^*) \hookrightarrow \hat{Y}$ is a Legendre one [Ar]. Indeed,

the contact distribution $D \subset T\hat{Y}$ at each point $\hat{y} \in \hat{Y}$ consists of those tangent vectors $V_{\hat{y}} \in T_{\hat{y}}\hat{Y}$ which satisfy the equation $\langle \hat{y}, \tau_*(V_{\hat{y}}) \rangle = 0$, where $\tau : \hat{Y} \rightarrow Y$ is a natural projection and the angular brackets denote the pairing of 1-forms and vectors at $\tau(\hat{y}) \in Y$. Since the submanifold $\hat{X} \subset \hat{Y}$ consists precisely of those projective classes of 1-forms in $\Omega^1 Y|_X$ which vanish when restricted on TX , we conclude that $T\hat{X} \subset D|_{\hat{X}}$. One may check that this association

$$\begin{array}{ccc} \text{Kodaira moduli space} & \longrightarrow & \text{Legendre moduli space} \\ \{X_t \hookrightarrow Y \mid t \in M\} & \longrightarrow & \left\{ \hat{X}_t := \mathbb{P}(N_{X_t|Y}^*) \hookrightarrow \hat{Y} := \mathbb{P}(\Omega^1 Y) \mid t \in M \right\} \end{array}$$

preserves completeness while changing its meaning, i.e. a *complete* Kodaira family of compact complex submanifolds is mapped into a *complete* family of compact complex Legendre submanifolds (which is usually not complete in the Kodaira sense).

The contact line bundle L on \hat{Y} is just the dual of the tautological line bundle $\mathcal{O}_{\hat{Y}}(-1)$. Simplifying the notations, $N := N_{X|Y}$ and $\hat{N} := N_{\hat{X}|\hat{Y}}$, we write down the following commutative diagram which explains how \hat{N} is related to $\rho^*(N)$ and L

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \rho^*(\Omega^1 X) \otimes L_{\hat{X}} & = & \rho^*(\Omega^1 X) \otimes L_{\hat{X}} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1 \hat{X} \otimes L_{\hat{X}} & \longrightarrow & \hat{N} & \longrightarrow & L_{\hat{X}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Omega_{\rho}^1 \otimes L_{\hat{X}} & \longrightarrow & \rho^*(N) & \longrightarrow & L_{\hat{X}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Here $L_{\hat{X}} = L|_{\hat{X}}$, ρ is a natural projection $\hat{X} \rightarrow X$, and Ω_{ρ}^1 is the bundle of ρ -vertical 1-forms, i.e. the dual of $T_{\rho} = \ker : T\hat{X} \rightarrow TX$. Using this diagram it is not hard to show that there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\rightarrow H^0(X, N \otimes S^2(N^*)) \rightarrow H^0\left(\hat{X}, L_{\hat{X}} \otimes S^2(\hat{N}^*)\right) \rightarrow H^0(X, N^* \otimes TX) \rightarrow \\ &\rightarrow H^1(X, N \otimes S^2(N^*)) \rightarrow H^1\left(\hat{X}, L_{\hat{X}} \otimes S^2(\hat{N}^*)\right) \rightarrow H^1(X, N^* \otimes TX) \rightarrow \dots \end{aligned}$$

Proposition 5 *Let $X \hookrightarrow Y$ be a compact complex rigid submanifold with rigid normal bundle N such that $H^1(X, N) = 0$ and let M be the associated Kodaira moduli space. If*

$$H^1(X, N \otimes S^2(N^*)) = H^1(X, N^* \otimes TX) = 0, \quad (4)$$

then the associated Kodaira moduli space M comes equipped with an induced 1-flat G -structure with the Lie algebra \mathfrak{g} of G being characterized by the following exact sequence of Lie algebras

$$0 \longrightarrow H^0(X, N \otimes N^*) \longrightarrow \mathfrak{g} \longrightarrow H^0(X, TX) \longrightarrow 0$$

Proof. By Kodaira theorem [K], there is a complete family, $\{X_t \hookrightarrow Y \mid t \in M\}$, of compact complex submanifolds and hence the associated complete family, $\{\hat{X}_t \hookrightarrow \hat{Y} \mid t \in M\}$, of compact complex Legendre submanifolds with $\hat{X}_t = \mathbb{P}(N_t^*)$ and $\hat{Y} = \mathbb{P}(\Omega^1 M)$. Equations (4) together with the above long exact sequence of cohomology groups imply the vanishing of $H^1(\hat{X}, L_{\hat{X}} \otimes S^2(\hat{N}^*))$. This together with Theorem 2(iii) and the subsequent Remark implies in turn that the induced G -structure on M is 1-flat. The final statement about the Lie algebra of G follows from Theorem 2(ii), the exact sequence

$$0 \longrightarrow L_{\hat{X}} \otimes \rho^*(N^*) \longrightarrow L_{\hat{X}} \otimes \hat{N}^* \longrightarrow \rho^*(TX) \longrightarrow 0$$

and the fact that $H^0(\hat{X}, L_{\hat{X}} \otimes \rho^*(N^*)) = H^0(X, N \otimes N^*)$. \square

If, for example, X is the projective line \mathbb{CP}^1 embedded into a complex manifold Y with normal bundle $N = \mathbb{C}^{2k} \otimes \mathcal{O}(1)$, $k \geq 1$, then by Proposition 5 the associated Kodaira moduli space M , which is a $4k$ -dimensional complex manifold, comes equipped canonically with a complexified quaternionic structure, in accordance with [Pe, P-P]. For other results on geometric structures induced on Kodaira moduli spaces we refer to [Me2, M-P].

7. Involutive G -structures. Let M be a complex m -dimensional manifold and let $\mathcal{G} \subset \mathcal{L}^* M$ be an irreducible holomorphic G -structure with reductive G (hence G must be isomorphic to G_s or $G_s \times \mathbb{C}^*$ for some semisimple Lie group $G_s \subset GL(m, \mathbb{C})$). Since \mathcal{G} is irreducible, there is a naturally associated subbundle $\tilde{F} \subset \Omega^1 M$ whose typical fiber is the cone in \mathbb{C}^m defined as the G_s -orbit of the line spanned by a highest weight vector. Denote $\tilde{\mathcal{F}} = \tilde{F} \setminus 0_{\tilde{F}}$, where $0_{\tilde{F}}$ is the "zero" section of $\tilde{p} : \tilde{F} \rightarrow M$ whose value at each $t \in M$ is the vertex of the cone $\tilde{p}^{-1}(t)$. The quotient bundle $\nu : \mathcal{F} = \tilde{\mathcal{F}}/\mathbb{C}^* \rightarrow M$ is then a subbundle of the projectivized cotangent bundle $\mathbb{P}(\Omega^1 M)$ whose fibres X_t are isomorphic to the generalised flag variety G_s/P , where P is the parabolic subgroup of G_s which preserves the highest weight vector in \mathbb{C}^m up to a scale. The total space of the cotangent bundle $\Omega^1 M$ has a canonical holomorphic symplectic 2-form ω which makes the sheaf of holomorphic functions on $\Omega^1 M$ into a sheaf of Lie algebras via the Poisson bracket $\{f, g\} = \omega^{-1}(df, dg)$.

Definition 6 *An irreducible G -structure $\mathcal{G} \rightarrow M$ is called involutive if $\tilde{\mathcal{F}}$ is a coisotropic submanifold of the symplectic manifold $\Omega^1 M \setminus 0_{\Omega^1 M}$.*

The first motivation behind this definition is the following

Lemma 7 *Every irreducible 1-flat G -structure is involutive.*

Proof. It is well known that a submanifold of a symplectic manifold is isotropic if and only if the associated ideal sheaf is the sheaf of Lie algebras relative to the Poisson bracket. This condition obviously holds for a locally flat G -structure. Since the Poisson bracket involves only a 1st order differential operator, this condition must also be satisfied for a 1-flat G -structure. \square

The pullback, $i^*\omega$, of the symplectic form ω from $\Omega^1 M \setminus 0_{\Omega^1 M}$ to its submanifold $i : \tilde{\mathcal{F}} \rightarrow \Omega^1 M \setminus 0_{\Omega^1 M}$ defines a distribution $\mathcal{D} \subset T\tilde{\mathcal{F}}$ as the kernel of the natural "lowering of indices" map $T\tilde{\mathcal{F}} \xrightarrow{i^*\omega} \Omega^1 \tilde{\mathcal{F}}$, i.e. $\mathcal{D}_e = \left\{ V \in T_e \tilde{\mathcal{F}} : V \lrcorner i^*\omega = 0 \right\}$ at each point $e \in \tilde{\mathcal{F}}$. Using the fact that $d(i^*\omega) = i^*d\omega = 0$, one can show that this distribution is integrable and

thus defines a foliation of $\tilde{\mathcal{F}}$ by holomorphic leaves. We shall assume from now on that the space of leaves, \tilde{Y} , is a complex manifold. This assumption imposes no restrictions on the local structure of M . The fact that the Lie derivative, $\mathcal{L}_V i^* \omega = V \lrcorner i^* d\omega + d(V \lrcorner i^* \omega) = 0$, vanishes for any vector field V tangent to the leaves implies that $i^* \omega$ is the pullback relative to the canonical projection $\tilde{\mu} : \tilde{\mathcal{F}} \rightarrow \tilde{Y}$ of a closed 2-form $\tilde{\omega}$ on \tilde{Y} . It is easy to check that $\tilde{\omega}$ is non-degenerate which means that $(\tilde{Y}, \tilde{\omega})$ is a symplectic manifold. There is a natural action of \mathbb{C}^* on $\tilde{\mathcal{F}}$ which leaves \mathcal{D} invariant and thus induces an action of \mathbb{C}^* on \tilde{Y} . The quotient $Y := \tilde{Y}/\mathbb{C}^*$ is an odd dimensional complex manifold which has a double fibration structure

$$Y \xleftarrow{\mu} \mathcal{F} = \tilde{\mathcal{F}}/\mathbb{C}^* \xrightarrow{\nu} M$$

and thus contains an analytic family of compact submanifolds $\{X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y \mid t \in M\}$ with $X_t = \tilde{X}_t/\mathbb{C}^* \simeq G_s/P$. Next, inverting a well-known procedure of symplectivisation of a contact manifold [Ar], it is not hard to show that Y has a complex contact structure such that all the submanifolds $X_t \hookrightarrow Y$ are isotropic. The contact line bundle L on Y is just the quotient $L = \tilde{\mathcal{F}} \times \mathbb{C}/\mathbb{C}^*$ relative to the natural multiplication map $\tilde{\mathcal{F}} \times \mathbb{C} \rightarrow \tilde{\mathcal{F}} \times \mathbb{C}$, $(p, c) \rightarrow (\lambda p, \lambda c)$, where $\lambda \in \mathbb{C}^*$. This can be summarized as follows.

Proposition 8 *Given an irreducible G -structure $\mathcal{G} \rightarrow M$ with reductive G . There is canonically associated a complex contact manifold (Y, L) containing a $\dim M$ -parameter family $\{X_t \hookrightarrow Y \mid t \in M\}$ of isotropically emdedded generalized flag varieties $X_t = G_s/P$, where G_s is the semisimple part of G and P is the parabolic subgroup of G_s leaving invariant a highest weight vector in the typical fibre of $\Omega^1 M \rightarrow M$ up to a scale.*

Let e be any point of $\tilde{\mathcal{F}} \subset \Omega^1 M \setminus 0_{\Omega^1 M}$. Restricting a "lowering of indices" map $T_e(\Omega^1 M) \xrightarrow{\lrcorner \omega} \Omega_e^1(\Omega^1 M)$ to the subspace \mathcal{D}_e , one obtains an injective map

$$0 \longrightarrow \mathcal{D}_e \xrightarrow{\lrcorner \omega} \mathcal{N}_e^*,$$

where \mathcal{N}_e^* is the fibre of the conormal bundle of $\tilde{\mathcal{F}} \hookrightarrow \Omega^1 M \setminus 0_{\Omega^1 M}$. Therefore, the rank of the distribution \mathcal{D} is equal at most to $\text{rank } \mathcal{N}^* = \dim M - \dim X_t - 1$. It is easy to check that $\text{rank } \mathcal{D}$ is maximal possible if and only if \mathcal{G} is involutive. In this case the contact manifold Y associated to \mathcal{G} has dimension

$$\begin{aligned} \dim Y &= \dim \tilde{Y} - 1 = \dim \tilde{\mathcal{F}} - \text{rank } \mathcal{D} - 1 \\ &= (\dim M + \dim X_t + 1) - (\dim M - \dim X_t - 1) - 1 = 2 \dim X_t + 1 \end{aligned}$$

which means that the associated complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ is an analytic family of compact Legendre submanifolds. This argument partly explains the following result

Theorem 9 *Let $G \subset GL(m, \mathbb{C})$ be an arbitrary connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups does not contain any of the groups H considered in Theorem 3(ii). If \mathcal{G} is any involutive $G \times \mathbb{C}^*$ -structure on an m -dimensional manifold M , then there exists a complex contact manifold (Y, L) and a Legendre submanifold $X \hookrightarrow Y$ with $X = G/P$ for some parabolic subgroup $P \subset G$ and with L_X being very ample, such that, at least locally, M is canonically isomorphic to the associated Legendre moduli space and $\mathcal{G} = \mathcal{G}_{ind}$.*

In the case of involutive G -structures $\mathcal{G} \rightarrow M$ with G as in Theorem 3(i) one can still canonically identify the base manifold M with a Legendre moduli space, but now \mathcal{G} is properly contained in \mathcal{G}_{ind} .

In conclusion, the earlier posed question — how large is the family of G -structures which can be constructed by twistor methods of Theorem 2? — has the following answer: this family consists of involutive G -structures.

8. Affine connections with "very little torsion". With any irreducible G -structure \mathcal{G} on a complex manifold M one can associate the *torsion number* defined as follows

$$l = \frac{1}{2}(\dim M - \dim G/P - \text{rank} \mathcal{D} - 1).$$

Here P is the parabolic subgroup of G leaving invariant up to a scale a highest weight vector in the typical fibre of TM , and \mathcal{D} is the distribution associated to \mathcal{G} as explained in section 7. We see that the torsion number l is composed of two very different parts: the first part, $\dim M - \dim G/P$, encodes only "linear" information about the particular irreducible representation of G , while the second part, $\text{rank} \mathcal{D} - 1$, measures how this particular representation is "attached" to the base manifold M . It is not difficult to prove that l is always a non-negative *integer*. This fact alone shows that the proposed combination l of four natural numbers does give some insight into the structure of \mathcal{G} . This impression can be further strengthened by the fact that l has a nice geometric interpretation. Remember that, by Proposition 8, the G -structure \mathcal{G} gives rise to a complex contact manifold (Y, L) and a family $\{X_t \hookrightarrow Y \mid t \in M\}$ of isotropic submanifolds parameterised by M . Then, in these terms,

$$l = \frac{1}{2}(\dim Y - 1) - \dim X_t,$$

i.e. l measures how much X_t lacks to be a Legendre submanifold.

Why is l called a *torsion* number? It is not difficult to show that the torsion number of any *1-flat* G -structure is zero. Therefore, a G -structure on M may have non-vanishing l only if it has a non-vanishing invariant torsion (but not vice versa, as we shall see in a moment). Moreover, the larger l is, the less integrable is the distribution \mathcal{D} and, in this sense, the "larger" is the invariant torsion.

Definition 10 *The torsion number of an affine connection ∇ is the torsion number of the associated holonomy bundle \mathcal{G}_∇ .*

Definition 11 *An affine connection with torsion is said to have very little torsion if its torsion number is zero.*

The class of affine connections with very little torsion is a sibling of the class torsion-free affine connections in the sense that both these classes (and only these two classes) have involutive holonomy bundles (it is not difficult to show that a G -structure has $l = 0$ if and only if it is involutive). This means in particular that *all* connections of both types can be constructed by twistor methods on appropriate Legendre moduli spaces. Another conclusion is that the class of affine connections with very little torsion is non empty — one can construct plenty of them using Legendre deformations problems " $X \hookrightarrow (Y, L)$ " such that $H^1(X, L_X \otimes S^2(J^1 L_X)^*) \neq 0$. This does not mean, however, that this class is

enormously large — on the contrary, as the table below shows, the list of irreducibly acting holonomies of affine connections with very little torsion must be very restricted.

Group G	The only possible holonomy representations of G in the class of connections with “very little torsion”	
	representation	dim
$SL(2, \mathbb{C})$	$\begin{smallmatrix} k \\ \bullet \end{smallmatrix} \quad k \geq 4$	$k(k+1)/2$
$SL(2, \mathbb{C})SL(2, \mathbb{C})$	$\begin{smallmatrix} 1 & k \\ \bullet & \otimes \bullet \end{smallmatrix} \quad k \geq 2$	$2k+2$
	$\begin{smallmatrix} 1 & k \\ \bullet & \otimes \bullet \end{smallmatrix} \quad k \geq 2$	$3k+3$
$SL(3, \mathbb{C})$	$\begin{smallmatrix} 1 & 1 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	8
	$\begin{smallmatrix} 1 & 2 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	15
$Sp(4, \mathbb{C})$	$\begin{smallmatrix} 1 & 1 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	16
	$\begin{smallmatrix} 2 & 0 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	14
	$\begin{smallmatrix} 3 & 0 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	30
G_2	$\begin{smallmatrix} 0 & 2 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	27
	$\begin{smallmatrix} 0 & 3 \\ \bullet & \text{---} \bullet \end{smallmatrix}$	77

Here irreducible representations are written in the notations of [B-E]).

In conclusion we note that the problem of classifying all irreducibly acting reductive holonomies of affine connections with zero or very little torsion has a strong purely symplectic flavour — it is nearly equivalent to the problem of classifying all generalized flag varieties X which can be realized as complex Legendre submanifolds of *non-trivial* contact manifolds Y (non-trivial in the sense that the germ of Y at X is *not* isomorphic to the germ of the total space of the jet bundle $J^1 L_X$, for some line bundle $L_X \rightarrow X$, at its zero section).

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